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On the decay of solutions in nonsimple elastic solids with memory

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ABSTRACT

The decay of solutions in nonsimple elasticity with memory is addressed, analyzing how the decay rate is influenced by the different dissipation mechanisms appearing in the equations. In particular, a first order dissipation is shown to guarantee the asymptotic stability of the related solution semigroup, but is not strong enough to entail exponential stability. The latter occurs for a dissipation mechanism of the second order, that is, the same order as the one of the leading operator.

1. Introduction

New mathematical models for continuum materials have been proposed in the last years, such as the one describing *nonsimple elastic solids*. These are particular elastic solids, introduced by Mindlin [11] and Toupin [18], among others, accounting for effects related to the size of defects and the microstructures. More details on the subject can be found in the books [1,8]. In this work, we focus on nonsimple elastic solids with memory, which exhibit a viscous dissipation mechanism of memory type. Existence, uniqueness and asymptotic stability of solutions arising from models of this kind have been proved in [17]. Here, we mean to deepen the study of the asymptotic behavior.

The constitutive equations of nonsimple elastic solids are known to contain first and second order gradients, both contributing to dissipation. It is then interesting to understand the relevance of the two different dissipation mechanisms which can appear in the theory. In particular, we would like to clarify the influence on the decay of each dissipation mechanism, studying the longterm dynamics when only one of them is present. In fact, the simultaneous presence of both mechanisms can be analyzed as well, with inessential changes in the proofs: In that situation, the behavior turns out to be the same as if only the higher order dissipation appears in the equations.

In order to simplify the analysis, we will treat the case of anti-plane deformations, leading to an easier version of the mechanical problem, where the general system is replaced by a single equation: namely, (2.5) or (2.6) below, according to the dissipation mechanism under consideration. Actually, we will study an abstract equation (see (3.1)), depending on a parameter $\vartheta \in [0,1]$, which contains (2.5) and (2.6) as particular instances, corresponding to $\vartheta = 1/2$ (first order dissipation), and $\vartheta = 1$ (second order dissipation), respectively. Roughly speaking, we can summarize our results as follows: Under suitable assumptions on the relaxation function, asymptotic stability takes place for every value of $\vartheta \in [0,1]$. However, the decay is not uniformly controlled by a decreasing exponential unless $\vartheta = 1$. In other words, exponential decay may occur only when the dissipation mechanism is of the same order as the one of the leading operator appearing in the equation.

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Plan of the paper

In the next Section 2, we introduce the linear equations of three-dimensional viscoelasticity, considering anti-plane shear deformations corresponding to the isotropic homogeneous case. This leads to the formulation of the abstract equation (2.4), of which (2.5) and (2.6) are two particular cases, obtained when only one of the dissipation mechanisms is present. Section 3 is devoted to the construction of the semigroup of solutions, whose asymptotic and exponential stability is discussed in the subsequent Section 4 and Section 5, respectively.

2. The model

We consider the linear theory of second order elastic bodies with memory. In the standard notation, the motion equations are

$$\rho \ddot{u}_i = (\tau_{ji} - \kappa_{kji,k})_{,j} + \rho f_i. \tag{2.1}$$

Here, $\{u_i\}$ is the displacement vector in the reference configuration, ρ is the mass density, $\{f_i\}$ is the body force, κ_{ijk} is the hyperstress tensor, and τ_{ij} is defined as

$$\tau_{ij} = t_{ij} + \kappa_{kij,k},$$

where t_{ij} is the stress tensor. If we restrict our attention to materials with a center of symmetry, the tensors of odd order vanish; accordingly, the constitutive equations become

$$\tau_{ij} = C_{ijkl}(0)u_{k,l} + \int_0^\infty C'_{ijkl}(s)u_{k,l}(t-s)\,\mathrm{d}s,$$

$$\kappa_{ijk} = K_{ijklmn}(0)u_{l,mn} + \int_0^\infty K'_{ijklmn}(s)u_{l,mn}(t-s)\,\mathrm{d}s,$$

where the *prime* denotes the derivative with respect to s. Plugging these equations into (2.1), and assuming the material homogeneous, we are led to the system

$$\begin{aligned}\rho \ddot{u}_i &= C_{ijkl}(0)u_{k,lj} - K_{ijklmn}(0)u_{l,mnkj}\\ &\quad + \int_0^\infty \big[C'_{ijkl}(s)u_{k,lj}(t-s) - K'_{ijklmn}(s)u_{l,mnkj}(t-s)\big]\,\mathrm{d}s + \rho f_i,\end{aligned}$$

which, in the case of isotropic materials, takes the form

$$\begin{aligned}\rho \ddot{u}_i &= \big(\lambda(0) + 2\mu(0)\big)u_{j,ji} + \mu(0)u_{i,jj} - \ell_1(0)u_{j,jkki} - \ell_2(0)u_{i,jjkk}\\ &\quad + \int_0^\infty \big[\big(\lambda'(s) + 2\mu'(s)\big)u_{j,ji}(t-s) + \mu'(s)u_{i,jj}(t-s)\big]\,\mathrm{d}s\\ &\quad - \int_0^\infty \big[\big(\ell'_1(s)u_{j,jkki}(t-s) + \ell'_2(s)\big)u_{i,jjkk}(t-s)\big]\,\mathrm{d}s + \rho f_i.\end{aligned} \tag{2.2}$$

For anti-plane shear deformations, i.e. when

$$u_1 = u_2 = 0, \qquad u_3 = u(x_1, x_2),$$

we have the equality $u_{j,j} = 0$. Hence, assuming null body forces, the first two equations of (2.2) are automatically satisfied, whereas the third one reads

$$\rho \ddot{u} = \mu(0)\Delta u - \ell_2(0)\Delta^2 u + \int_0^\infty \big[\mu'(s)\Delta u(t-s) - \ell'_2(s)\Delta^2 u(t-s)\big]\,\mathrm{d}s. \tag{2.3}$$

In this paper, we study the initial–boundary value problem related to (2.3), with particular reference to the asymptotic behavior of solutions. To this end, we consider a bounded domain $\Omega \subset \mathbb{R}^2$ with smooth boundary $\partial\Omega$, and we require u to satisfy the homogeneous boundary conditions

$$u(\boldsymbol{x}, t) = \Delta u(\boldsymbol{x}, t) = 0, \quad (\boldsymbol{x}, t) \in \partial\Omega \times (0, \infty),$$

and the initial condition at time $t = 0$

$$u(\boldsymbol{x}, -s) = g(\boldsymbol{x}, s), \quad (\boldsymbol{x}, s) \in \Omega \times [0, \infty),$$

for some assigned function g, playing the role of an initial datum. Introducing the positive operator $A = -\Delta$ with homogeneous Dirichlet boundary conditions, (2.3) is given the abstract form

$$\rho\ddot{u} + \mu(\infty)Au + \ell_2(\infty)A^2u + \int_0^\infty \mu'(s)A\big[u(t) - u(t-s)\big]\,\mathrm{d}s + \int_0^\infty \ell_2'(s)A^2\big[u(t) - u(t-s)\big]\,\mathrm{d}s = 0. \tag{2.4}$$

We will restrict the analysis to the case where either μ or ℓ_2 (but not both) are constant functions. This amounts to considering the following two particular instances of Eq. (2.4):

$$\rho\ddot{u} + \mu(\infty)Au + \ell_2(\infty)A^2u + \int_0^\infty \mu'(s)A\big[u(t) - u(t-s)\big]\,\mathrm{d}s = 0, \tag{2.5}$$

$$\rho\ddot{u} + \mu(\infty)Au + \ell_2(\infty)A^2u + \int_0^\infty \ell_2'(s)A^2\big[u(t) - u(t-s)\big]\,\mathrm{d}s = 0. \tag{2.6}$$

3. The semigroup of solutions

3.1. The equation

Setting for simplicity all the physical constants equal to 1, we write Eqs. (2.5)–(2.6) in the unitary fashion

$$\ddot{u}(t) + A\left[u(t) + Au(t) + \int_0^\infty \kappa(s)A^{2\vartheta-1}\eta^t(s)\,\mathrm{d}s\right] = 0, \tag{3.1}$$

for $\vartheta \in \{1/2, 1\}$, with the positions

$$\eta^t(s) = u(t) - u(t-s)$$

and

$$\kappa(s) = \begin{cases} \mu'(s) & \text{if } \vartheta = 1/2, \\ \ell_2'(s) & \text{if } \vartheta = 1. \end{cases}$$

Actually, we will consider a generalized version of (3.1), allowing the parameter ϑ to run in the whole interval $[0, 1]$. Our preliminary task is to describe the solutions to Eq. (3.1) in terms of a semigroup acting on some Hilbert space. This is accomplished within the Dafermos' history framework [3], viewing $\eta^t(s)$ as an auxiliary variable, accounting for the past history of u, which (formally) fulfills the "boundary-value" problem

$$\partial_t\eta^t(s) = -\partial_s\eta^t(s) + \partial_t u(t), \quad \eta^t(0) = 0.$$

To this end, a suitable functional setting is needed.

3.2. Assumptions and notation

Given a strictly positive selfadjoint linear operator A on a real Hilbert space $(H, \langle\cdot,\cdot\rangle, \|\cdot\|)$ with domain $\mathrm{dom}(A) \Subset H$, we denote by $\{\alpha_n\}$ the (strictly positive) increasing sequence of eigenvalues of A, and by $\{w_n\}$ the corresponding eigenvectors. Moreover, for $r \in \mathbb{R}$, we consider the scale of Hilbert spaces

$$H^r = \mathrm{dom}\big(A^{r/2}\big), \qquad \langle u, v\rangle_r = \big\langle A^{r/2}u, A^{r/2}v\big\rangle.$$

The *memory kernel* $\kappa : \mathbb{R}^+ = (0, \infty) \to [0, \infty)$ is assumed to be a nonincreasing absolutely continuous summable function of total mass

$$\int_0^\infty \kappa(s)\,\mathrm{d}s = m_\kappa > 0.$$

For a *fixed* $\vartheta \in [0, 1]$, we introduce the L^2-weighted space

$$\mathcal{M} = L^2_\kappa\big(\mathbb{R}^+; H^{2\vartheta}\big), \qquad \langle\eta, \psi\rangle_{\mathcal{M}} = \int_0^\infty \kappa(s)\big\langle\eta(s), \psi(s)\big\rangle_{2\vartheta}\,\mathrm{d}s,$$

along with the infinitesimal generator T of the right-translation semigroup on $\mathcal{M}$

$$T\eta = -\eta', \quad \operatorname{dom}(T) = \Big\{\eta \in \mathcal{M}\colon\ \eta' \in \mathcal{M},\ \lim_{s\to 0}\big\|\eta(s)\big\|_{2\vartheta} = 0\Big\},$$

where the *prime* stands for distributional derivative, recalling the basic equality (cf. [7])

$$2\langle T\eta, \eta\rangle_{\mathcal{M}} = \int_0^\infty \kappa'(s)\big\|\eta(s)\big\|_{2\vartheta}^2\,\mathrm{d}s, \quad \forall \eta \in \operatorname{dom}(T). \tag{3.2}$$

Finally, we define the product Hilbert space

$$\mathcal{H} = H^2 \times H \times \mathcal{M},$$

normed by

$$\big\|(u, v, \eta)\big\|_{\mathcal{H}}^2 = \|u\|_2^2 + \|u\|_1^2 + \|v\|^2 + \|\eta\|_{\mathcal{M}}^2.$$

3.3. The contraction semigroup

Introducing the vector $\zeta(t) = (u(t), v(t), \eta^t)$, we view Eq. (3.1) as the ODE in $\mathcal{H}$

$$\frac{\mathrm{d}}{\mathrm{d}t}\zeta(t) = \mathbb{A}\zeta(t), \tag{3.3}$$

where the linear operator $\mathbb{A}$ is defined as

$$\mathbb{A}(u, v, \eta) = \left(v, -A\left[u + Au + \int_0^\infty \kappa(s)A^{2\vartheta-1}\eta(s)\,\mathrm{d}s\right], T\eta + v\right),$$

with domain

$$\operatorname{dom}(\mathbb{A}) = \left\{(u, v, \eta) \in H^2 \times H^2 \times \operatorname{dom}(T)\colon\ Au + \int_0^\infty \kappa(s)A^{2\vartheta-1}\eta(s)\,\mathrm{d}s \in H^2\right\}.$$

We address the reader to the survey paper [7] for more details on the equivalence between the formulations (3.1) and (3.3).

An application of the classical Lumer–Phillips theorem [15] yields the following result.

Theorem 3.1. *The operator $\mathbb{A}$ is the infinitesimal generator of a contraction semigroup*

$$S(t) = \mathrm{e}^{t\mathbb{A}} : \mathcal{H} \to \mathcal{H}.$$

Besides, for every $t > 0$ and every $z \in \operatorname{dom}(\mathbb{A})$, the energy equality

$$\big\|S(t)z\big\|_{\mathcal{H}}^2 = \|z\|_{\mathcal{H}}^2 + \int_0^t \left(\int_0^\infty \kappa'(s)\big\|\eta^\tau(s)\big\|_{2\vartheta}^2\,\mathrm{d}s\right)\mathrm{d}\tau \tag{3.4}$$

holds, where η^t is the third component of the solution $\zeta(t) = S(t)z$ to (3.3) *with initial datum $\zeta(0) = z$.*

Finally, for every $z = (u_0, v_0, \eta_0) \in \mathcal{H}$, the third component η^t of the solution has the explicit representation formula (see [7])

$$\eta^t(s) = \begin{cases} u(t) - u(t-s), & s \leqslant t, \\ \eta_0(s-t) + u(t) - u_0, & s > t. \end{cases} \tag{3.5}$$

4. Asymptotic stability

As a first step towards the longterm analysis of the semigroup, we discuss the asymptotic stability of $S(t)$ on $\mathcal{H}$ as $\vartheta \in [0, 1]$.

Theorem 4.1. *For every fixed $\vartheta \in [0, 1]$, the semigroup $S(t)$ is asymptotically stable*; *namely,*

$$\lim_{t\to\infty}\big\|S(t)z\big\|_{\mathcal{H}} = 0, \quad \forall z \in \mathcal{H}.$$

Proof. Due to a general result on contraction semigroups (see e.g. [6, Appendix]), the claim is established upon showing that

(i) the equality $\|S(t)z\|_{\mathcal{H}} = \|z\|_{\mathcal{H}}$ for all $t > 0$ implies that $z = 0$;
(ii) the set $\bigcup_{t>0} S(t)z$ is relatively compact in $\mathcal{H}$ whenever $z \in \mathfrak{D}$, for some dense subset $\mathfrak{D}$ of $\mathcal{H}$.

In order to prove (i), let $z = (u_0, v_0, \eta_0) \in \mathcal{H}$ be such that

$$\left\|S(t)z\right\|_{\mathcal{H}} = \left\|\left(u(t), v(t), \eta^t\right)\right\|_{\mathcal{H}} = \|z\|_{\mathcal{H}}, \quad \forall t > 0, \tag{4.1}$$

and choose a sequence $z_n \in \mathrm{dom}(A)$, $z_n \to z$. For $t > 0$ arbitrarily fixed, we have

$$\left\|S(t)z_n\right\|_{\mathcal{H}} = \left\|\left(u_n(t), v_n(t), \eta_n^t\right)\right\|_{\mathcal{H}} \to \left\|S(t)z\right\|_{\mathcal{H}}$$

and, up to a subsequence,

$$\left\|\eta_n^t(s)\right\|_{2\vartheta} \to \left\|\eta^t(s)\right\|_{2\vartheta}, \quad \text{for a.e. } s \in \mathbb{R}^+.$$

Exploiting (3.4) and the Fatou lemma, we obtain

$$\begin{aligned}
0 = \lim_{n\to\infty}\left(\left\|S(t)z_n\right\|_{\mathcal{H}}^2 - \|z_n\|_{\mathcal{H}}^2\right) &= \lim_{n\to\infty}\int_0^t\left(\int_0^\infty \kappa'(s)\left\|\eta_n^\tau(s)\right\|_{2\vartheta}^2\,\mathrm{d}s\right)\mathrm{d}\tau \\
&\leqslant \int_0^t\left(\int_0^\infty \kappa'(s)\left\|\eta^\tau(s)\right\|_{2\vartheta}^2\,\mathrm{d}s\right)\mathrm{d}\tau \leqslant 0,
\end{aligned}$$

which forces the equality $\|\eta^t(s)\|_{2\vartheta} = 0$ for almost every $t > 0$ and almost every s in the set $\mathcal{O} = \{s \in \mathbb{R}^+ \colon \kappa'(s) < 0\}$. Then, from the representation formula (3.5), we learn that $u(t) = u(t-s)$, for almost every $t > 0$ and almost every $s \in (0,t) \cap \mathcal{O}$. Since $\mathcal{O}$ has positive measure and the map $t \mapsto \|u(t)\|_{2\vartheta}$ is continuous, we easily conclude that, for every $t > 0$,

$$u(t) = u_0 \quad \Rightarrow \quad \dot{u}(t) = v(t) = v_0 = 0.$$

Accordingly, denoting as usual by χ the characteristic function, (3.5) turns into

$$\eta^t(s) = \chi_{(t,\infty)}(s)\eta_0(s-t),$$

and by (4.1) we infer the equality

$$\|\eta_0\|_{\mathcal{M}}^2 = \left\|\eta^t\right\|_{\mathcal{M}}^2 = \int_0^\infty \kappa(s+t)\left\|\eta_0(s)\right\|_{2\vartheta}^2\,\mathrm{d}s, \quad \forall t > 0.$$

Letting $t \to \infty$, the dominated convergence theorem gives

$$\eta^t = \eta_0 = 0, \quad \forall t > 0.$$

To finish the proof, we are left to show the equality $u_0 = 0$. But this is an immediate consequence of Eq. (3.3), whose second component now reads

$$A[u_0 + Au_0] = 0.$$

Concerning (ii), we introduce the product Hilbert space

$$\mathcal{V} = H^4 \times H^2 \times L_\kappa^2\left(\mathbb{R}^+; H^{2\vartheta+2}\right),$$

and we define the dense subset $\mathfrak{D}$ of $\mathcal{H}$ as

$$\mathfrak{D} = \left\{(u, v, \eta) \in \mathcal{V} \colon \eta \in \mathrm{dom}(T),\ \sup_{s>0}\left\|\eta(s)\right\|_{2\vartheta} < \infty\right\}.$$

In particular, by the same arguments of the previous section, $S(t)$ turns out to be a contraction semigroup on $\mathcal{V}$ as well, so that

$$\left\|S(t)z\right\|_{\mathcal{V}} \leqslant \|z\|_{\mathcal{V}}, \quad \forall z \in \mathcal{V}.$$

Let now $z = (u_0, v_0, \eta_0) \in \mathfrak{D}$ be fixed. Setting $S(t)z = (u(t), v(t), \eta^t)$, the above inequality implies that

$$\sup_{t>0}\left\|u(t)\right\|_4 < \infty, \qquad \sup_{t>0}\left\|v(t)\right\|_2 < \infty,$$

which, together with the compact embedding $H^4 \times H^2 \Subset H^2 \times H$, ensure the relative compactness in $H^2 \times H$ of the set $\bigcup_{t>0}(u(t), v(t))$. By the same token,

$$\sup_{t>0} \|\eta^t\|_{L^2_\kappa(\mathbb{R}^+;H^{2\vartheta+2})} < \infty.$$

Besides, taking advantage of (3.5), it is readily seen that

$$\sup_{t>0} \|T\eta^t\|_{\mathcal{M}} < \infty, \qquad \sup_{t>0} \sup_{s>0} \|\eta^t(s)\|_{2\vartheta} < \infty.$$

In light of [14, Lemma 5.5], the latter bounds allow us to conclude that $\bigcup_{t>0} \eta^t$ is relatively compact in $\mathcal{M}$, so establishing (ii). □

5. Exponential stability

5.1. Statement of the result

We now turn our attention to the more interesting issue of exponential stability. Recall that $S(t)$ is said to be *exponentially stable* if there exist constants $M \geqslant 1$ and $\varkappa > 0$ such that

$$\|S(t)z\|_{\mathcal{H}} \leqslant M\mathrm{e}^{-\varkappa t}\|z\|_{\mathcal{H}}, \quad \forall z \in \mathcal{H}.$$

We will see that, in presence of a weakly singular kernel κ, exponential stability cannot occur if $\vartheta < 1$. Conversely, if $\vartheta = 1$ and κ fulfills a suitable decay property, the semigroup is exponentially stable.

Theorem 5.1. *The following hold.*

(i) *$S(t)$ is not exponentially stable if $\vartheta < 1$ and κ satisfies the relation*

$$\lim_{s\to 0} s^{1-\vartheta}\kappa(s) = 0. \tag{5.1}$$

(ii) *$S(t)$ is exponentially stable if $\vartheta = 1$ and κ satisfies the relation*

$$\kappa'(s) + \delta\kappa(s) \leqslant 0, \tag{5.2}$$

for some $\delta > 0$ and almost every $s \in \mathbb{R}^+$.

Remark 5.2. Condition (5.2) has been widely used to prove exponential stability in several problems with memory (e.g. [3,4,6,9,10,12]). Nonetheless, the exponential decay of $S(t)$ can be shown to hold under weaker assumptions on κ (see [13]).

The proof of the theorem is based on the next abstract lemma stated in [5], which is actually an alternative version of a classical result (cf. [2,16]).

Lemma 5.3. *The contraction semigroup $S(t) = \mathrm{e}^{t\mathbb{A}}$ on $\mathcal{H}$ is exponentially stable if and only if there exists $\varepsilon > 0$ such that*

$$\inf_{\lambda\in\mathbb{R}} \|(\mathrm{i}\lambda - \mathbb{A})z\|_{\mathcal{H}} \geqslant \varepsilon\|z\|_{\mathcal{H}}, \quad \forall z \in \mathrm{dom}(\mathbb{A}). \tag{5.3}$$

The space $\mathcal{H}$ appearing in (5.3) is understood to be the complexification of the original real Hilbert space $\mathcal{H}$.

5.2. Proof of Theorem 5.1(i)

The following generalization of a result from [6] will be needed.

Lemma 5.4. *Let* (5.1) *hold for some $\vartheta \in [0, 1)$. Then,*

$$\lim_{\lambda\to\infty} \lambda^\vartheta \int_0^\infty \kappa(s)\mathrm{e}^{-\mathrm{i}\lambda s}\,\mathrm{d}s = 0.$$

Proof. We have the equality

$$\lambda^\vartheta \int_0^\infty \kappa(s)\mathrm{e}^{-\mathrm{i}\lambda s}\,\mathrm{d}s = \mathfrak{I}_1 + \mathfrak{I}_2,$$

where

$$\mathfrak{I}_1 = \lambda^\vartheta \int_0^{\pi/\lambda} \kappa(s) e^{-i\lambda s}\, ds + \frac{\lambda^\vartheta}{2} \int_{\pi/\lambda}^{2\pi/\lambda} \kappa(s) e^{-i\lambda s}\, ds,$$

$$\mathfrak{I}_2 = -\frac{\lambda^\vartheta}{2} \int_{\pi/\lambda}^{\infty} e^{-i\lambda s} \left[\int_s^{s+\pi/\lambda} \kappa'(\sigma)\, d\sigma \right] ds.$$

Setting $\nu(s) = s^{1-\vartheta}\kappa(s)$, the first term is estimated by

$$|\mathfrak{I}_1| \leqslant \lambda^\vartheta \int_0^{2\pi/\lambda} \kappa(s)\, ds \leqslant \lambda^\vartheta \sup_{s<2\pi/\lambda} \nu(s) \int_0^{2\pi/\lambda} \frac{1}{s^{1-\vartheta}}\, ds \leqslant \frac{(2\pi)^\vartheta}{\vartheta} \sup_{s<2\pi/\lambda} \nu(s),$$

whereas for the second one, exchanging the order of integration, we get

$$|\mathfrak{I}_2| \leqslant \lambda^\vartheta \int_{\pi/\lambda}^{\infty} \left[\int_{s+\pi/\lambda}^{s} \kappa'(\sigma)\, d\sigma \right] ds \leqslant \pi^\vartheta \nu(\pi/\lambda).$$

It is apparent that both terms tend to zero as $\lambda \to \infty$.

Remark 5.5. It is worth noting that (5.1) cannot be weakened. Let us show this for the kernel

$$\kappa(s) = \begin{cases} s^{-1+\vartheta}, & s \leqslant 1, \\ e^{-(s-1)}, & s > 1, \end{cases}$$

for which $\nu(s) = 1$ as $s \leqslant 1$. By direct calculations,

$$\int_0^\infty \kappa(s) e^{-i\lambda s}\, ds = \int_0^\lambda \frac{e^{-i\sigma}}{\sigma^{1-\vartheta}}\, d\sigma + \frac{\lambda^\vartheta e^{-i\lambda}}{1 + i\lambda}.$$

The latter term goes to zero as $\lambda \to \infty$. On the other hand,

$$\Im\left(\int_0^\lambda \frac{e^{-i\sigma}}{\sigma^{1-\vartheta}}\, d\sigma \right) = -\int_0^\lambda \frac{\sin\sigma}{\sigma^{1-\vartheta}}\, d\sigma,$$

which has a (strictly) negative limit as $\lambda \to \infty$.

Our claim will follow by showing that condition (5.3) fails to hold. Arguing as in [6], for any $n \in \mathbb{N}$, we introduce the vector

$$\zeta_n = \left(0, 0, \alpha_n^{-\vartheta} w_n\right) \in \mathcal{H}, \qquad \|\zeta_n\|_{\mathcal{H}} = \sqrt{m_\kappa}.$$

For $n \in \mathbb{N}$ and $\lambda > 0$, we consider the equation in the unknown variable $z_n = (u_n, v_n, \eta_n)$

$$(i\lambda - \mathbb{A}) z_n = \zeta_n,$$

looking for a solution of the form

$$u_n = p_n w_n, \qquad v_n = q_n w_n, \qquad \eta_n = \varphi_n w_n,$$

for some $p_n, q_n \in \mathbb{C}$ and $\varphi_n \in L^2_k(\mathbb{R}^+)$ with $\varphi_n(0) = 0$. Hence, we arrive at the system

$$\begin{cases} i\lambda p_n - q_n = 0, \\ i\lambda q_n + \alpha_n p_n + \alpha_n^2 p_n + \alpha_n^{2\vartheta} \displaystyle\int_0^\infty \kappa(s) \varphi_n(s)\, ds = 0, \\ i\lambda \varphi_n(s) + \varphi_n'(s) - q_n = \alpha_n^{-\vartheta}. \end{cases}$$

An integration of the third equation yields

$$\varphi(s) = \frac{1}{i\lambda}\left(q_n + \alpha_n^{-\vartheta}\right)\left(1 - e^{-i\lambda s}\right),$$

and substituting φ in the second equation, we obtain

$$q_n\big(\lambda^2-\alpha_n-\alpha_n^2-\alpha_n^{2\vartheta}m_\kappa\big)+q_n\alpha_n^{2\vartheta}\int_0^\infty \kappa(s)\mathrm{e}^{-\mathrm{i}\lambda s}\,\mathrm{d}s=\alpha_n^{\vartheta}\left(m_\kappa-\int_0^\infty \kappa(s)\mathrm{e}^{-\mathrm{i}\lambda s}\,\mathrm{d}s\right).$$

Setting

$$\lambda_n=\sqrt{\alpha_n+\alpha_n^2+\alpha_n^{2\vartheta}m_\kappa},$$

and choosing $\lambda=\lambda_n$, we finally get

$$q_n=\left(\frac{\lambda_n}{\alpha_n}\right)^{\vartheta}\frac{m_\kappa-c_n}{\lambda_n^{\vartheta}c_n}\sim\frac{m_\kappa-c_n}{\lambda_n^{\vartheta}c_n},$$

where we defined

$$c_n=\int_0^\infty \kappa(s)\mathrm{e}^{-\mathrm{i}\lambda_n s}\,\mathrm{d}s.$$

In light of Lemma 5.4, we conclude that

$$\lim_{n\to\infty}c_n=\lim_{n\to\infty}\lambda_n^{\vartheta}c_n=0.$$

Thus,

$$\|z_n\|_{\mathcal{H}}\geqslant\|v_n\|=|q_n|\to\infty,$$

violating (5.3).

5.3. Proof of Theorem 5.1(ii)

By contradiction, assume the conclusion false. Then, Lemma 5.3 ensures the existence of sequences $\lambda_n\in\mathbb{R}$ and $z_n=(u_n,v_n,\eta_n)\in\operatorname{dom}(\mathbb{A})$, satisfying the bound

$$\|z_n\|_{\mathcal{H}}^2=\|u_n\|_2^2+\|u_n\|_1^2+\|v_n\|^2+\|\eta_n\|_{\mathcal{M}}^2=1, \tag{5.4}$$

such that

$$\lim_{n\to\infty}\big\|(\mathrm{i}\lambda_n-\mathbb{A})z_n\big\|_{\mathcal{H}}=0. \tag{5.5}$$

In particular, exploiting (3.2),

$$\int_0^\infty \kappa'(s)\big\|\eta_n(s)\big\|_2^2\,\mathrm{d}s=2\Re\langle T\eta_n,\eta_n\rangle_{\mathcal{M}}=2\Re\big\langle(\mathrm{i}\lambda-\mathbb{A})z_n,z_n\big\rangle_{\mathcal{H}}\to 0.$$

Therefore, we infer from (5.2) that

$$\|\eta_n\|_{\mathcal{M}}^2\leqslant-\frac{1}{\delta}\int_0^\infty \kappa'(s)\big\|\eta_n(s)\big\|_2^2\,\mathrm{d}s\to 0. \tag{5.6}$$

Writing (5.5) in components, we get

$$\mathrm{i}\lambda_n u_n-v_n\to 0\quad\text{in } H^2, \tag{5.7}$$

$$\mathrm{i}\lambda_n v_n+A\left[Au_n+u_n+\int_0^\infty \kappa(s)\eta_n(s)\,\mathrm{d}s\right]\to 0\quad\text{in } H, \tag{5.8}$$

$$\mathrm{i}\lambda_n\eta_n-T\eta_n-v_n\to 0\quad\text{in } \mathcal{M}. \tag{5.9}$$

Since, by (5.6),

$$\big|\langle\eta_n,u_n\rangle_{\mathcal{M}}\big|\leqslant\int_0^\infty \kappa(s)\big\|\eta_n(s)\big\|_2\,\mathrm{d}s\leqslant\sqrt{m_\kappa}\,\|\eta_n\|_{\mathcal{M}}\to 0,$$

adding (5.7) times v_n and (5.8) times u_n, and taking the real part, we find the convergence

$$\|u_n\|_2^2 + \|u_n\|_1^2 - \|v_n\|^2 \to 0.$$

In view of (5.4) and (5.6), we conclude that

$$\|v_n\| \to 1/\sqrt{2}.$$

We will reach a contradiction by showing that, at the same time,

$$\|v_n\| \to 0. \tag{5.10}$$

To this end, select $s_\star > 0$ such that $\kappa(s_\star) > 0$. Defining the auxiliary kernel

$$\omega(s) = \chi_{[0,s_\star)}(s)\kappa(s_\star) + \chi_{[s_\star,\infty)}(s)\kappa(s),$$

we consider the L^2-weighted space

$$\mathcal{W} = L^2_\omega(\mathbb{R}^+; H^2),$$

continuously embedded into $\mathcal{M}$. Then, from (5.9), we obtain the limit

$$\mathrm{i}\lambda_n\langle\eta_n, A^{-2}v_n\rangle_{\mathcal{W}} - \langle T\eta_n, A^{-2}v_n\rangle_{\mathcal{W}} - \langle v_n, A^{-2}v_n\rangle_{\mathcal{W}} \to 0. \tag{5.11}$$

We preliminarily observe that

$$\sup_{n\in\mathbb{N}} |\lambda_n|\|v_n\|_{-2} < \infty,$$

which is a straightforward consequence of (5.8). Hence, recalling (5.6),

$$\left|\lambda_n\langle\eta_n, A^{-2}v_n\rangle_{\mathcal{W}}\right| \leqslant |\lambda_n|\|v_n\|_{-2}\int_0^\infty \omega(s)\left\|\eta_n(s)\right\|_2 \mathrm{d}s \leqslant \sqrt{m_\kappa}\,|\lambda_n|\|v_n\|_{-2}\|\eta_n\|_{\mathcal{M}} \to 0.$$

Concerning the second term, an integration by parts yields

$$\langle T\eta_n, A^{-2}v_n\rangle_{\mathcal{W}} = -\int_0^\infty \omega(s)\frac{\mathrm{d}}{\mathrm{d}s}\langle\eta_n(s), A^{-2}v_n\rangle_2 \,\mathrm{d}s = \int_{s_\star}^\infty \kappa'(s)\langle\eta_n(s), A^{-2}v_n\rangle_2 \,\mathrm{d}s,$$

where the boundary terms vanish by standard arguments (cf. [7]). Thus, by (5.6),

$$\begin{aligned}\left|\langle T\eta_n, A^{-2}v_n\rangle_{\mathcal{W}}\right| &\leqslant -\|v_n\|\int_{s_\star}^\infty \kappa'(s)\left\|\eta_n(s)\right\|_2 \mathrm{d}s \\ &\leqslant \left(-\kappa(s_\star)\int_0^\infty \kappa'(s)\left\|\eta_n(s)\right\|_2^2 \mathrm{d}s\right)^{1/2} \to 0.\end{aligned}$$

In summary, (5.11) reduces to

$$\langle v_n, A^{-2}v_n\rangle_{\mathcal{W}} = \|v_n\|\left(\int_0^\infty \omega(s)\,\mathrm{d}s\right) \to 0,$$

yielding the desired convergence (5.10).

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